Inversion of all Principal Submatrices of a Matrix

Let $A_m$ be an $m \times m$ principal submatrix of an infinite-dimensional matrix $A$. We give a simple formula which expresses $A_{m+1}^{-1}$ in terms of $A_m^{-1}$, and based on this formula, an algorithm which computes the inverses of $A_m$ for $m = 1, 2, 3, \ldots, n$ using only $2n^2 - 2n + n$ arithmetic operations. This is an improvement over the naive method of computing the inverses separately which would require $\sum_{m=1}^{n} m^3 = O(n^4)$ arithmetic operations.

I. THE MOTIVATION

The following problem is frequently encountered in many aerospace engineering calculations, such as the Kalman filtering algorithm, as well as in signal processing and systems control theory [2, 3, 1, 4]. Given a tolerance $\epsilon > 0$ and an infinite-dimensional linear system of equations of the form

$$
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & x_1 \\
    a_{21} & a_{22} & a_{23} & \cdots & x_2 \\
    a_{31} & a_{32} & a_{33} & \cdots & x_3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & x_n
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots \\
    b_n
\end{bmatrix}
$$

find a finite-dimensional solution vector $x_n = [x_1, x_2, \ldots, x_n]^T$ such that

$$f(x_n) \leq \epsilon$$

for some nonnegative objective functional $f$. This can be achieved by solving the following sequence of (increasing) finite-dimensional problems:

$$
\begin{bmatrix}
    a_{11} \\
    a_{21} \\
    a_{31} \\
    \vdots \\
    a_{n1}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots \\
    b_n
\end{bmatrix}
$$

and checking if the above tolerance criterion is satisfied. Solving this sequence of finite-dimensional linear systems is equivalent to computing the following sequence of matrix inversions:

$$A_1^{-1}, A_2^{-1}, A_3^{-1}, \ldots, A_n^{-1}$$

where $A_m$ for $1 \leq m \leq n$ is an $m \times m$ principal submatrix of the infinite-dimensional system matrix $A = [a_{ij}]_{i,j=1}^{\infty}$.

II. THE FORMULA

We partition the $(m+1) \times (m+1)$ matrix $A_{m+1}$ as follows

$$
A_{m+1} = 
\begin{bmatrix}
    a_{11} & \cdots & a_{1m} & a_{1,m+1} \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & \cdots & a_{mm} & a_{m,m+1} \\
    a_{m+1,1} & \cdots & a_{m+1,m} & a_{m+1,m+1}
\end{bmatrix}
$$

We assume that $A_m^{-1}$ exists and

$$a := a - c^T A_m^{-1} b \neq 0.$$ 

Let $\alpha := 1/d$. We claim that the inverse of $A_{m+1}$ is given by

$$A_{m+1}^{-1} = 
\begin{bmatrix}
    A_m^{-1} & \alpha A_m^{-1} b c^T A_m^{-1} - \alpha A_m^{-1} b \\
    -c^T A_m^{-1} & \alpha
\end{bmatrix}
$$

This claim is easily verified as follows:

$$A_{m+1} A_{m+1}^{-1} = 
\begin{bmatrix}
    A_m & b \\
    c^T & a
\end{bmatrix}
\begin{bmatrix}
    A_m^{-1} & \alpha A_m^{-1} b c^T A_m^{-1} - \alpha A_m^{-1} b \\
    -c^T A_m^{-1} & \alpha
\end{bmatrix}
= 
\begin{bmatrix}
    I_m & 0 \\
    0 & I 
\end{bmatrix} = I_{m+1}.$$
TABLE I

Computation of $A_{n+1}$ Given $A_n$

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(A_{m}^{-1})b$</td>
<td>$m(2m-1)$</td>
</tr>
<tr>
<td>2.</td>
<td>$(c^T A_{m}^{-1}c)$</td>
<td>$m(2m-1)$</td>
</tr>
<tr>
<td>3.</td>
<td>$(c^T A_{m}^{-1}b)$</td>
<td>$2m-1$</td>
</tr>
<tr>
<td>4.</td>
<td>$(a - c^T A_{m}^{-1}b)$</td>
<td>$1$</td>
</tr>
<tr>
<td>5.</td>
<td>$\alpha = 1/(a - c^T A_{m}^{-1}b)$</td>
<td>$1$</td>
</tr>
<tr>
<td>6.</td>
<td>$(-\alpha)(a_{m}^{-1}b)$</td>
<td>$m$</td>
</tr>
<tr>
<td>7.</td>
<td>$(-\alpha)(c^T A_{m}^{-1}c)$</td>
<td>$m$</td>
</tr>
<tr>
<td>8.</td>
<td>$(a_{m}^{-1}b)(c^T A_{m}^{-1})$</td>
<td>$m^2$</td>
</tr>
<tr>
<td>9.</td>
<td>$(a_{m}^{-1} + (a_{m}^{-1}b)c^T A_{m}^{-1})$</td>
<td>$m^2$</td>
</tr>
</tbody>
</table>

It is also straightforward to show that $A_{m+1}^{-1} A_{m+1} = I_{m+1}$.

III. THE ALGORITHM

The algorithm computes the inverses in the following order: $A_1^{-1}, A_2^{-1}, A_3^{-1}, \ldots, A_{n-1}^{-1}$. Once $A_1^{-1}$ is obtained, we compute $A_2^{-1}$ using some additional vector-vector, vector-matrix, and matrix-matrix operations. We give the steps of the algorithm in Table I, where the number of arithmetic operations required at each step is also indicated.

Let $T(m)$ be the number of arithmetic operations required to compute the inverse of $A_m$. Assuming that $A_1^{-1}$ is already computed using $T(m)$ arithmetic operations, we proceed to compute $A_2^{-1}$ using additional $6m^2 + 2m + 1$ arithmetic operations, as can be seen from Table I. Note that $A_1$ is only a scalar, thus $T(1) = 1$. As $m$ runs from 2 to $n$, we obtain the inverses of all $A_m$ for $2 \leq m \leq n$ using

$$T(n) = T(n - 1) + 6(n - 1)^2 + 2(n - 1) + 1$$

arithmetic operations. Thus, the number of arithmetic operations required for computing $A_n^{-1}$ for $1 \leq m \leq n$ is found as

$$T(n) = T(n - 1) + 6(n - 1)^2 + 2(n - 1) + 1$$

$$= T(n - 2) + 6(n - 2)^2 + 6(n - 1)^2$$

$$+ 2(n - 2) + 2(n - 1) + 1 + 1$$

$$\vdots$$

$$= T(1) + 6 \sum_{m=1}^{n-1} m^2 + 2 \sum_{m=1}^{n-1} m + \sum_{m=1}^{n-1} 1$$

$$= 1 + (n - 1)n(2n - 1) + (n - 1)n + (n - 1)$$

$$= 2n^3 - 2n^2 + n.$$